Geometric Monopoles:
Clifford Algebra and Magnetic Monopoles

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Abstract : Just enough of a taste of Clifford algebra is presented to reformulate via Clifford algebra Maxwell's Equations and the Maxwell's Equations with magnetic monopoles. This paper is geared at students in Physics 137: String Theory; however, only elementary electrodynamics, basic vector algebra, and a dash of bravery is required. Thus, classical monopole theories are reviewed along the way. An appendix with further relevant details that may not be completely necessary for a full development of the paper topics and plentiful references are included.

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1. Introduction

"The laws of the universe are written in the language of mathematics."
—The first line of my high school differential equations book

The most powerful available mathematics system to date is Clifford algebra. Yet, the essence of this math is far from known; some hold the misconception that its usage is confined to quantum mechanics, namely, having to do with the Pauli and Dirac matrices, while others use it without realizing it. Much of the physics community holds it as an esoteric math necessary only for the dauntless math-hybrid theorist at the forefronts of the most cutting-edge of research. This paper hopes to dispel this "belief."

Briefly, Clifford algebra is an essential tool for various theories. In QED, many algebraic manipulations involving Dirac matrices are carried out more efficiently without reference to any particular matrix representation; the method to achive this is via Clifford algebra. In Kaluza-Klein and dimension normalization theories, one must deal with spaces of arbitrary dimension; no longer can the usual 4×4 representation of Dirac matrices be used, and avoiding reference to any particular dimension requires Clifford algebra. Furthermore, Clifford algebra is essential for superstring theories. Via the geometric language of Clifford algebra, many concepts in physics are clarified, unified, and extended in often surprising directions. Specifically, Clifford algebra eliminates the formal gaps that traditionally separate classical, quantum, and relativistic physics, thereby suggesting resolutions to the major physics problem of the twentith century: quantum gravity.

In the more familiar special relativity, Clifford algebra can be used to derive all the usual relations such as the Lorentz transformations and velocity addition formulae — but in a unified fashion, as opposed to the bumbling arithmetic it is usually presented in. In the old mechanics, Clifford algebra provides a term that distinguishes cross product entities such as angular momentum from regular vectors; indeed, such is necessary as pseudovectors reflect differently than regular vectors. In electrodynamics, as will be shown, Clifford algebra wields the power to express all of Maxwell's Equations even in non-vacuums via just one equation, thereby exceeding both differential forms and tensor analysis in elegance "potential." In quantum mechanics, Clifford algebra lends to some interesting non-standard interpretations.[1]

Quintessentially, Clifford algebra encompasses both the familiar hackneyed vector calculus and less familiar differential forms. The structure of differential forms and tangent vectors is embedded in the structure of Clifford algebra with only slight modifications; moreover, the formalism of those can be substantially simplified in the context of Clifford algebra.[2]

[1] Just for fun: http://www.rialian.com/rnboyd/mind-matter-unification.htm

[2] $\vec{dx}'$ and ∂_j are presented in the usual formulation as coordinate bases of dual spaces, distinction between these spaces being necessary when no metric is given. Most physics applications require the introduction of a non-singular metric generating an isomorphism between the spaces, essentially rendering them identical. The formalism of differential forms neglects this, maintaining

This paper does not aim at the needless task of writing a full primer on Clifford algebra. Unlike other mathematics, Clifford algebra is rather simple-minded, as many applications can arise from knowing just a tiny bit of it. *Instead, this paper aims at providing a bare minimal introduction to Clifford algebra in the context of its basic applications to Maxwell's Equations and magnetic monopoles.*

The audience is to be non-monopole-specialist students of the course Physics 137: String Theory, and thus the classic monopole theories will be reviewed along the way. An appendix expounds on topics merely skimmed through in the context of the paper in an overall informal fashion, so as not to (further) intimidate the reader. To maintain brevity, much will be left unsaid, deferred to a preliminary list of references provided to quench the reader's curiosity.

The author has only recently been initiated into the allure of Clifford algebra, and already, it has become a permanent addiction. Perhaps the reader will find Clifford algebra equally if not more interesting.

2. A Glimpse of Clifford Algebra

Consider the following 2-D vectors:

$$a = p\mathbf{e_1} + q\mathbf{e_2} \tag{2.1}$$

$$b = s\mathbf{e_1} + t\mathbf{e_2} \tag{2.2}$$

where a and b are vectors, while $\mathbf{e_i}$ denote orthonormal bases.

Ignoring the usual rules of vector arithmetic, treating the terms as terms from elementary arithmetic, the product ab becomes:

$$ab = (p\mathbf{e_1} + q\mathbf{e_2})(s\mathbf{e_1} + t\mathbf{e_2}) \tag{2.3}$$

$$= ps\mathbf{e_1}\mathbf{e_1} + pt\mathbf{e_1}\mathbf{e_2} + qs\mathbf{e_2}\mathbf{e_1} + qt\mathbf{e_2}\mathbf{e_2} \tag{2.4}$$

$$= ps + qt + (pt - qs)\mathbf{e_{12}} \tag{2.5}$$

where in going from (2.4) to (2.5), the scalar components were taken to commute and the following substitutions were made:

$$\mathbf{e_1}\mathbf{e_2} = \mathbf{e_{12}} = -\mathbf{e_2}\mathbf{e_1} = -\mathbf{e_{21}} \tag{2.6}$$

$$\mathbf{e_i}\mathbf{e_i} = \mathbf{e_i}^2 = 1 \tag{2.7}$$

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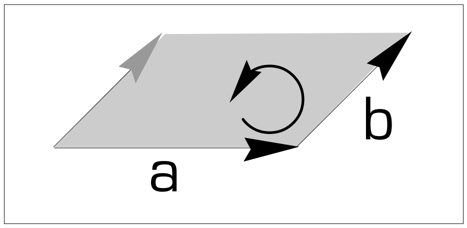
unnecessary distinctions, creating a multitude of products, mappings, and spaces requiring some seriously bulky notation. The tangent vector, ∂_j, in the usual formalism, is treated as the image of the differential form $g_{ji}\vec{dx}^i$ under an isomorphic mapping; in Clifford algebra, the analog becomes $\gamma_j = g_{\hat{j}i}\gamma^i$

ab portrays the essence of the geometric product in $\mathcal{C}l(2)$.[3] It is a sum of a scalar component reminiscent of the dot product and a $\mathbf{e_1e_2}$ component. The $\mathbf{e_1e_2}$ term is a bivector. It is denoted as $a \wedge b = (pt - qs)\mathbf{e_{12}}$



The wedge product of a and b forms a directed plane.

The bivector $a \wedge b$ is formed by "wedging" vector b along vector a. As seen in the figure above, b starts at the location of the gray vector and is wedged along a to its final black vector location. Although it can be seen as the generalized analog of a cross product, the wedge product can be easily generalized into higher dimensions. Note the sense of the product.

Written via this new terminology, the geometric product is represented as:

$$ab = a \cdot b + a \wedge b \tag{2.8}$$

The geometric product illustrates the key feature of Clifford Algebra; it allows one to add and multiply quantities of different dimensions. Note that it obeys the usual rules of an algebra:[4] it is associative, distributive, closed.

It can easily be generalized into bivectors, i.e., denoting B as a bivector:

$$aB = a \cdot B + a \wedge B \tag{2.9}$$

The proof above shall be omitted for brevity; see any of the references. A key feature of this product involves a term that's lower in dimension and a term that is higher in dimension.

One can define trivectors and multivectors of n dimensions, in general, by wedging the vector a_i n times:[5]

$$\psi = a_1 \wedge a_2 \wedge \cdots a_{n-1} \wedge a_n \tag{2.10}$$

Recall the last term in (2.5) and its association as the vector a wedged with the vector b:

$$(pt - qs)\mathbf{e_1e_2} = a \wedge b \tag{2.11}$$

From that, one can deduce the Rosetta stone between the usual Maxwell's Equations (ME) and ME's via Clifford algebra, i.e., one can assert a duality relation

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[3] $\mathcal{C}l(2)$ denotes the Clifford Algebra Space generated by two vectors. Its bases are c_0 (scalar), e_1, e_2 (vector), and e_1e_2 (bivector). This notation is similar to the familiar $\mathcal{C}(2)$ denoting the 2-dimensional complex space... except $\mathcal{C}l(2)$ obeys different rules, and only the ones relevant to the topic are talked about in this paper. See the Appendix or the References for more details.

[4] See Appendix

[5] For a start on further details on geometric algebra, see the the Appendix and the References

between the usual cross product and the wedge product by multiplying by the pseudoscalar $I = \mathbf{e_1 e_2 e_3}$:[6]

$$
\begin{aligned}
I a \times b &= I(pt - qs)\mathbf{e_3} && (2.12)\\
&= (pt - qs)\mathbf{e_1 e_2 e_3 e_3} && (2.13)\\
&= (pt - qs)\mathbf{e_1 e_2} && (2.14)\\
I a \times b &= a \wedge b && (2.15)
\end{aligned}
$$

One property of the pseudoscalar is that it commutes in odd dimensions and anticommutes in even dimensions. Consider the $3-D$ pseudoscalar: $e_1 e_2 e_3$. Suppose one were to multiply it with an arbitrary vector. For the purpose of an explicit calculation, say this vector is e_1. Thus $Ie_1 = e_1 e_2 e_3 e_1 = -e_1 e_2 e_1 e_3 = e_1 e_1 e_2 e_3 = e_1 I \Rightarrow Ie_1 = e_1 I$. As another example, the $4-D$ pseudoscalar: $Ie_1 = e_1 e_2 e_3 e_4$ right multiplied with e_1 yields $e_1 e_2 e_3 e_4 e_1 = -e_1 e_2 e_3 e_1 e_4 = e_1 e_2 e_1 e_3 e_4 = -e_1 e_1 e_2 e_3 e_4 = -e_1 I \Rightarrow Ie_1 = -e_1 I$.

This concludes the bit of bare necessary Clifford algebra required to understand this paper. The reader is encouarged to peruse the Appendix and References for further details.

3. (Classical) Monopoles

There is probably that distinct distaste associated with your memory of the usual Maxwell's Equations (ME): there's an uncanny asymmetry in them. How might they be made more symmetric? Consider the free-space source-free MEs:

$$
\begin{aligned}
\nabla \cdot \vec{E} &= 0 & \nabla \cdot \vec{B} &= 0\\
-\nabla \times \vec{E} &= \partial_t \vec{B} & \nabla \times \vec{B} &= \partial_t \vec{E}
\end{aligned} \qquad (3.1)
$$

If all the $\vec{E}$ were changed to $-\vec{B}$ and all the $\vec{B}$ were changed to $\vec{E}$, the exact same equations would arise! Similarly, if all $\vec{B}$ were changed to $-\vec{E}$ and all the $\vec{E}$ were changed to $\vec{B}$, one would end up with the same equations. This is the essence of electromagnetic duality. You get precisely the same theory if you interchange the electric and magnetic fields.

For example, suppose you have one theory that describes everything in terms of $\vec{E}$ and $\vec{B}$. If you make the duality substitutions described in the paragraph above, you would get the same theory. Also, since the energy density only depends on the square of the fields, it also stays invariant:

$$
u \propto \vec{E}^2 + \vec{B}^2 = (-\vec{B})^2 + \vec{E}^2 = (-\vec{E})^2 + \vec{B}^2 \qquad (3.2)
$$

[6] The pseudoscalar is the highest grade term in an algebra. In this case, the pseudoscalar has been implicitly generalized from $\mathcal{C}l2$ to $\mathcal{C}l3$; thus there are three bases vectors in our pseudoscalar - which has the same properties as the usual imaginary number i, viz., $i^2 = -1$.

However, as it stands, this duality only exists in source-free vacuums. Introduction of the magnetic monopole is necessary for inducing this duality in general:

$$\begin{array}{rclrcl} \nabla\cdot\vec{D} & = & \rho_e & \nabla\cdot\vec{B} & = & \rho_m \\ -\nabla\times\vec{E} & = & \partial_t\vec{B}+\vec{J_m} & \nabla\times\vec{H} & = & \partial_t\vec{D}+\vec{J_e} \end{array} \tag{3.3}$$

Incidentally, the following charge-current dualities arise: $(\rho, J) \to (\rho_m, J_m)$ and $(\rho_m, J_m) \to -(\rho, J)$.

4. Geometric Monopoles

The heartbreaking near but not quite symmetric form of ME can be ameliorated by ME for magnetic monopoles (MEM),[7] where the arrows denote ordinary space-vectors and $\vec{D} = \epsilon_0\vec{E} + P$ and $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$:

$$\begin{array}{rclrcl} \nabla\cdot\vec{D} & = & \rho_e & \nabla\cdot\vec{B} & = & \rho_m \\ -\nabla\times\vec{E} & = & \partial_t\vec{B}+\vec{J_m} & \nabla\times\vec{H} & = & \partial_t\vec{D}+\vec{J_e} \end{array} \tag{4.1}$$

In free space, reverting to natural units ($\epsilon_0 = \mu_0 = 1 \Rightarrow \frac{1}{\sqrt{\epsilon_0\mu_0}} = c = 1$), the above becomes:

$$\begin{array}{rclrcl} \nabla\cdot\vec{E} & = & \rho_e & \nabla\cdot\vec{B} & = & \rho_m \\ -\nabla\times\vec{E} & = & \partial_t\vec{B}+\vec{J_m} & \nabla\times\vec{B} & = & \partial_t\vec{E}+\vec{J_e} \end{array} \tag{4.2}$$

Via the duality transformation deduced in (2.15), the above becomes:

$$\begin{array}{rclrcl} \nabla\cdot\vec{E} & = & \rho_e & \nabla\cdot\vec{B} & = & \rho_m \\ -\nabla\wedge\vec{E} & = & I(\partial_t\vec{B}+\vec{J_m}) & \nabla\wedge\vec{B} & = & I(\partial_t\vec{E}+\vec{J_e}) \end{array} \tag{4.3}$$

Writing out the geometric product of $\nabla\vec{E}$ and $\nabla\vec{B}$ and substituting (4.3) yields:

$$\nabla\vec{E} = \nabla\cdot\vec{E} + \nabla\wedge\vec{E} = \rho_e - I\partial_t\vec{B} - I\vec{J_m} \tag{4.4}$$

$$\nabla\vec{B} = \nabla\cdot\vec{B} + \nabla\wedge\vec{B} = \rho_m + I\partial_t\vec{E} + I\vec{J_e} \tag{4.5}$$

$$\nabla I\vec{B} = I\rho_m - \partial_t\vec{E} - \vec{J_e} \tag{4.6}$$

The pseudoscalar is left-multiplied into the last geometric product above so that the fields can be compacted into the Faraday bivector. Define the Faraday bivector, the analogue of the covariant electromagnetic field strength tensor in Clifford algebra, to be:

[7] See Chapter 7 of Griffiths. *Introduction to Electrodynamics.*

$$F = \vec{E} + I\vec{B} \tag{4.7}$$

Thus, MEM becomes:

$$\nabla F + \partial_t F = \rho_e - \vec{J_e} + I\rho_m - I\vec{J_m} = \rho_e - \vec{J_e} + I(\rho_m - \vec{J_m}) \tag{4.8}$$

The above can be rendered into a more elegant form via recastment into manifestly Lorentz covariant form.

Define $\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = diag(+---)$ where $\mu, \nu \in (0,3)$ and define the spacetime currents J_e and J_m to have the following properties:

$$\begin{array}{rclcrcl} \rho_e & = & J_e \cdot \gamma_0 & & \vec{J_e} & = & J_e \wedge \gamma_0 \\ \rho_m & = & J_m \cdot \gamma_0 & & \vec{J_m} & = & J_m \wedge \gamma_0 \end{array} \tag{4.9}$$

Note $\gamma_0^2 = 1$ as implied by the metric. Thus, the following geometric products form:

$$\begin{array}{rclcl} \gamma_0 J_e & = & \gamma_0 \cdot J_e + \gamma_0 \wedge J_e & = & \rho_e - \vec{J_e} \\ \gamma_0 J_m & = & \gamma_0 \cdot J_m + \gamma_0 \wedge J_m & = & \rho_m - \vec{J_m} \\ J_m \gamma_0 & = & \gamma_0 \cdot J_m + J_m \wedge \gamma_0 & = & \rho_m + \vec{J_m} \end{array} \tag{4.10}$$

Note $\gamma_0 \nabla = \partial_t + \nabla$ (where the covariant del operator can be distinguished from the spatial del operator from context). Thus the left hand side of (4.8) becomes $\gamma_0 \nabla F$ and after substituting (4.10) the right hand side becomes:

$$\rho_e - \vec{J_e} + I(\rho_m - \vec{J_m}) = \gamma_0 J_e + I\gamma_0 J_m \tag{4.11}$$

$$= \gamma_0 J_e - \gamma_0 I J_m \tag{4.12}$$

$$= \gamma_0 J_e + \gamma_0 J_m I \tag{4.13}$$

Where the last lines follow since the pseudoscalar anticommutes in spacetime.

From the last of (4.10), one gets $J_m \gamma_0 = \rho_m + J_m$ Thus, $J_m = (\rho_m + \vec{J_m})\gamma_0$.

Left multiplying both sides of (4.8) with γ_0 after substituting the covariant form of the del operator on the left side and the right side with (4.13), the manifestly covariant MEM becomes:

$$\nabla F = J_e + J_m I \tag{4.14}$$

This is a fantastic result! Via Clifford algebra, the MEM becomes a mere one-liner...

Yet, now is a good point to bring up an interesting caveat. In the *absense* of magnetic monopoles, as will be proven in the appendix, Clifford analysis gives $\nabla F = J$ — an even more elegant equation! It seems that the need to insert a purely theoretical "aesthetic" term into Maxwell's Equations (thereby creating the "need" for monopoles) is merely a short-coming of ordinary vector analysis.

However, there is another reason for wishing for the existence of monopoles, if not for equational aesthetics. In the Dirac Theory, at least, they elegantly explain for quantized charge.

5. Dirac Quantization

From (4.2), one can derive the magnetic field of a point magnetic charge g.

$$\vec{B} = \frac{g}{r^2}\hat{r} \tag{5.1}$$

The magnetic flux is thus:

$$\Phi = \vec{B} \cdot d\vec{a} = B4\pi r^2 = 4\pi g \tag{5.2}$$

The wave function of a free particle is:

$$\Psi = \psi_0 \exp(\frac{i}{\hbar}(\vec{p} \cdot \vec{r} - Et)) \tag{5.3}$$

Its corresponding free particle Lagrangian would be $\mathcal{L} = \frac{1}{2}mv^2$.

In an electromagnetic field, denoting $\vec{A}$ as the magnetic potential, $\vec{v}$ as the particle velocity, and ϕ as the electric potential, the particle would be subject to the following interaction term in the Lagrangian:

$$\mathcal{L} = \frac{q}{c}\vec{A} \cdot \vec{v} - q\phi \tag{5.4}$$

The change in momentum is thus:

$$\vec{p}_{int} = q\vec{A}/c \tag{5.5}$$

Plugging this $\vec{p}$ back into (5.3), one gets the following phase change:

$$\Delta\delta = \frac{q}{\hbar c}\vec{A} \cdot \vec{r} = q\vec{A} \cdot \vec{r} \tag{5.6}$$

Where in arriving at the final equation, natural units were employed, i.e., $\hbar = c = 1$.

Consier a loop of fixed radius r and polar angle θ around the monopole. The total phase change is:

$$\Delta\delta = q\oint \vec{A} \cdot d\vec{l} \tag{5.7}$$

$$= q\int \nabla \times \vec{A} \cdot d\vec{a} \tag{5.8}$$

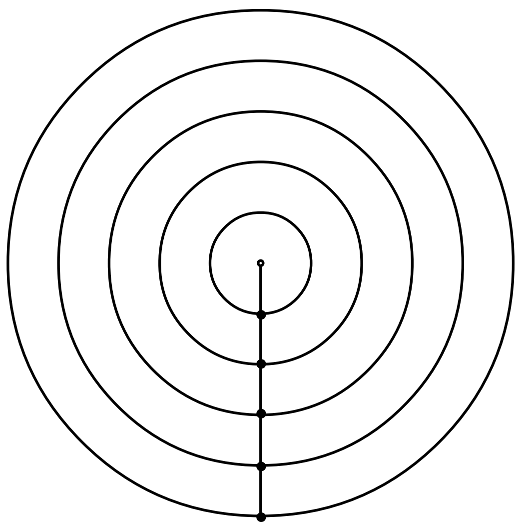
$$= q\int \vec{B} \cdot d\vec{a} = q\Phi(r,\theta) \tag{5.9}$$

By Stoke's Theorem and making the usual substitution of $\vec{B} = \nabla \times \vec{A}$, it becomes apparent that the total phase change around a loop is proportional to the magnetic flux.

But beware, for a singularity lurks above! At the poles of the sphere enclosing the monopole, the loop vanishes, shrinking into a point. At $\theta = 0$, the equations above hold without any further modifications, since the flux is zero through a surface of zero area. However, at $\theta = \pi$, the flux should approach that of (5.2), as the

Gaussian surface would be a complete spherical shell. Since at $\theta = \pi$, $d\vec{l} = 0$ (the loop shrinks to a point), $\vec{A}$ must be singular at this point in order to account for the non-vanishing flux.

Now, this analysis is taken at a fixed value of r, say $r = r_0$. Suppose another value of r is chosen, say $r = r_1$. At the coordinate point $(r = r_1, \theta = \pi, \phi)$, a singularity must be induced in $\vec{A}$ again to account for the nonzero flux. With any value of r, such a process of singularity introduction must occur, lest the equations do not hold to explain for the non-vanishing flux.



In this induced God's eye view, the singularities are marked with dots, while the ray connecting them (line, in spherical coordinates) is the Dirac String.

Note that in actuality, there is an uncountable infinity between each spherical shell, hence there is actually an infinite number of singularities along the Dirac String, hence forming the Dirac String. Only a few representative spheres are drawn.

Since $r \in (0, \infty)$, $\vec{A}$ must be singular for all points of the form $(r, \theta = \pi, \phi)$. Location-wise, the singularity exists in the "south pole" of the sphere. Consider an infinite ensemble of spherical shells, each of a different radius, spanning $r \in (0, \infty)$. The shells accumulate, filling the volume of all space. Because there is a singularity at the south pole of each shell, from an infinite ensemble of such, a ray of singularities starting from the south pole of the $r = 0$ shell evolves. Check out the singularity zoom-in diagram, where for the sake of properly illustrating the above concept, spaces have been integrated between uncountable infinities, essentially invoking a sort of God's eye view.

This ray is the Dirac String. Its invention came from asserting the flux by resorting to a singularity at $\theta = \pi$. For Ψ to be single valued, i.e., $\Psi(\theta) = \Psi(\theta + 2\pi)$, the phase change must be a multiple of 2π:

$$q\Phi = q4\pi g = 2\pi n \Rightarrow qg = \frac{n}{2} \tag{5.10}$$

This is the Dirac Quantization condition, explaining for the quantization of electric charge. Note that this equality requires that the electric and magnetic charge be inversely proportional, simultaneously allowing for one to replace the q's in Coulomb's Law with g's in something that might aptly be called $g - q$ coupling.

6. Conclusion

What was introduced in this (mainly) expository paper does not even begin to utilize the power of Clifford algebra. For one, its prudence in distinguishig vectors from pseudovetors/axial vectors[8] is completely omitted. Furthermore, its power as a rotation operator is also omitted. The vast majority of the perks mentioned in the Introduction are not expounded upon. Instead, only a relatively trivial application of the algebra in reworking Maxwell's Equations with monopoles is focused upon. Yet, there is an old rule of thumb that states that if one can get something out of a handicapped version of a mathematics, then the mathematics is definitely *extremely* powerful. Clifford algebra is, thus, definitely extremely powerful. The author hopes the paper has achieved its fundamental purpose in generating further interest on the applications of Clifford algebra in physics.

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[7] hang, Y. "The Real Monopole: A Physical and Metaphysical Glimpse." Physics 100C Paper (SP03).[9]

[8]The cross product is an example of a pseudovector. In three-dimensions, it is usually considered just another vector, yet its nature is fundamentally different than that of a vector. This can be seen under reflection. Suppose the vectors a and b are reflected across the yz plane. The x component would be negated. If you cross those two reflected vectors together, you would get a cross product that has its second and third components negated relative to the cross product composed of the unreflected vectors.

[9]Last year, when I first started reading up on monopoles, an idea came to me that the empirical observation of "curl fields/divergence-less fields" that imply the equation $\nabla \cdot \vec{B} = 0$ might be because the fields seen in d =3 is merely a projection of a higher dimensional divergence. However, from some of the math in Physics 137, it seems like my notion of the projection of a higher dimensional divergence above isn't so easily generalizable (from the notion of such in 3d). Yet, I am curious how such a notion of "projection" can be further expounded upon - might there be some dimension, say $d = x$, such that its divergence appears exactly like the curl in d=3?

7. Appendix

7.1. **From ME to $\nabla F = J$.** These are the usual Maxwell Eqs (ME) in SI. Note $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ and $\vec{H} = \vec{B}/\mu_0 - \vec{M}$.

$$\begin{array}{rclrcl} \nabla \cdot \vec{D} & = & \rho_e & \nabla \cdot \vec{B} & = & 0 \\ -\nabla \times \vec{E} & = & \partial_t \vec{B} & \nabla \times \vec{H} & = & \partial_t \vec{D} + \vec{J}_e \end{array} \tag{7.1}$$

In free space, the above becomes (in natural units):

$$\begin{array}{rclrcl} \nabla \cdot \vec{E} & = & \rho_e & \nabla \cdot \vec{B} & = & 0 \\ -\nabla \times \vec{E} & = & \partial_t \vec{B} & \nabla \times \vec{B} & = & \partial_t \vec{E} + \vec{J}_e \end{array} \tag{7.2}$$

There is an unsettling asymmetry in their current form. However, via (2.15), they are transformed into:

$$\begin{array}{rclrcl} \nabla \cdot \vec{E} & = & \rho_e & \nabla \cdot \vec{B} & = & 0 \\ -\nabla \wedge \vec{E} & = & \partial_t I \vec{B} & \nabla \wedge \vec{B} & = & I(\partial_t \vec{E} + \vec{J}_e) \end{array} \tag{7.3}$$

From the above, the geometric products of $\nabla \vec{E}$ and $\nabla I \vec{B}$ become:

$$\nabla \vec{E} = \nabla \cdot \vec{E} + \nabla \wedge \vec{E} = \rho - \partial_t I \vec{B} \tag{7.4}$$

$$\nabla I \vec{B} = \nabla \cdot \vec{E} + \nabla \wedge \vec{E} = -\vec{J} - \partial_t \vec{E} \tag{7.5}$$

Via the Faraday bivector (4.7), the sum of the above becomes:

$$\nabla F + \partial_t F = \rho - \vec{J} \tag{7.6}$$

The subscript e can be neglected since there is no such thing as magnetic current in the usual ME.

Using the properties of the spacetime currents, defined in (4.9), as well as the covariant form of the del operator (to wit: $\partial_t + \nabla = \nabla$), one gets:

$$\nabla F = J \tag{7.7}$$

Note that this equation is, indeed, more elegant than the MEM equation. Perhaps, presentation *is* everything.

7.2. **Algebra in 3-lines.** An algebra consists of a vector space V over a field F (e.g., $\mathcal{R}$ or $\mathcal{C}$) with a binary operation such that $\forall \alpha \in F$ and $\{A, B, C\} \in V$:

(1) $(\alpha A)B = A(\alpha B) = \alpha(AB)$
(2) $(A + B)D = AD + BD$ and $D(A + B) = DA + DB$
(3) $ABC = (AB)C = A(BC)$

7.3. **Clifford Algebra in a Nutshell.** A Clifford algebra for any space is generated from a set of Dirac matrices. Denote $\eta_{jk} = e_j \cdot e_k$ as the signature matrix (where in regular Euclidean space, it reduces to a mere Kronecker delta). Denote the set $\{\gamma_i\}$ as an orthonormal system of Dirac matrices having the following properties:

(1) $\hat{\gamma_j}\hat{\gamma_k} + \hat{\gamma_k}\hat{\gamma_j} = 2\eta_{jk} I$
(2) by taking all possible products of the n Dirac matries, one can form a set of 2^n linearly independent matrices.

7.4. **More on Multivectors.** The bases of $\mathcal{C}l_n$ (products of n-dimensional vectors) consist of $\frac{n!}{m!(n-m)!}$ m-vector basis. (This formula can be obtained by induction by expanding vectors in a basis $\{e_i\}$, multiplying out, etc.)

For example: $\mathcal{C}l_2 \Rightarrow n = 2$

$$\begin{array}{llll} m = 0 & \frac{2!}{0!2!} = 1 & \Rightarrow & \textit{scalar } c_0 \\ m = 1 & \frac{2!}{1!1!} = 2 & \Rightarrow & \textit{vector bases } e_1 \textit{ and } e_2 \\ m = 2 & \frac{2!}{2!0!} = 1 & \Rightarrow & \textit{bivector } e_1 e_2 \end{array} \tag{7.8}$$

$$\frac{n!}{m!(m-n)!} = \binom{n}{m} \Rightarrow \sum m = 0^n \binom{n}{m} = (1+1)^n = 2^n = d \tag{7.9}$$

The full linear space is thus 2^n dimensional.

As a check for $n = 2$, $d = 2^2 = 4$. The full bases of $\mathcal{C}l_2$ consisting of four elements $\{c_0, e_1, e_2, e_1e_2\}$ verifies this. For fun, the reader may wish to verify that the above formulae imply for $\mathcal{C}l_3$ the basis consists of the 2^3 elements:

$\{c_0, e_1, e_2, e_3, e_1e_2, e_2e_3, e_3e_1, e_1e_2e_3\}$.